Spin^c Structures on Manifolds and Geometric Applications

Roger NAKAD

April 10, 2012

Max Planck Institute for Mathematics
Vivatsgasse 7, 53111 Bonn
Germany

E-mail: nakad@mpim-bonn.mpg.de

Abstract

In this mini-course, we make use of Spin^c geometry to study special hypersurfaces. For this, we begin by selecting basic facts about Spin^c structures and the Dirac operator on Riemannian manifolds and their hypersurfaces. We end by giving a Lawson type correspondence for constant mean curvature surfaces in some 3-dimensional Thurston geometries.

Contents

1 Introduction and motivations

Having a Spin or Spin^c structure on a Riemannian manifold (M^n, g), we can define a natural first order elliptic differential operator called *the Dirac operator*. It acts on *spinor fields*: sections of a complex vector bundle ΣM called *the spinor bundle*. The geometry and topology of a Riemannian Spin or Spin^c manifold and their submanifolds are strongly related to the spectral properties of this operator.

On a compact Riemannian Spin manifold (M^n, g) of positive scalar curvature, A. Lichnerowicz [Lich63] proved that any eigenvalue λ of the Dirac operator D satisfies

$$\lambda^2 > \frac{1}{4} \inf_M \mathrm{Scal},$$

where Scal denotes the scalar curvature of (M^n, g). Then, the kernel of the Dirac operator is trivial and by the Atiyah-Singer theorem, the topological index of M^n is zero. This yields a topological obstruction for the existence of positive scalar metrics. Th. Friedrich [Fri80] refined the argument of A. Lichnerowicz and proved that

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M \mathrm{Scal}.$$

The equality case is characterized by the existence of a *real Killing spinor.* The existence of such spinors leads to restrictions on the manifold. For example, the manifold is Einstein and in dimension 4, it has constant sectional curvature. The classification of simply connected Riemannian Spin manifolds carrying real Killing spinors [Bär93] gives, in some dimensions, other examples than the sphere. These examples are relevant to physicists in general relativity where the Dirac operator plays a central role.

From an extrinsic point of view, Th. Friedrich [Fri98] characterised simply connected surfaces isometrically immersed in $\mathbb{R}^3$ by the existence of a spinor field satisfying the Dirac equation. Indeed,

$$\left\{ \begin{array}{c} (M^2, g) \hookrightarrow \mathbb{R}^3 \\ \text{of mean curvature } H \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} M^2 \text{ is a simply connected Spin surface} \\ \text{carrying a spinor field } \varphi \text{ of constant norm} \\ \text{satisfying} \quad \underbrace{D\varphi = H\varphi}_{\text{The Dirac equation}} \end{array} \right\}.$$

The spinor field φ is the restriction to the surface M of a parallel spinor on $\mathbb{R}^3$. A similar result holds for surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$ [Mor05]. As an application, we have an elementary proof of *a Lawson type correspondence*. H.B. Lawson proved a correspondence between surfaces of constant mean curvature in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$: *every simply connected minimal surface in $\mathbb{S}^3$ (resp. in $\mathbb{R}^3$) is isometric to a simply connected surface in $\mathbb{R}^3$ (resp. $\mathbb{H}^3$) with constant mean curvature equal to 1*. In 2001, O. Hijazi, S. Montiel and X. Zhang [HMZ01a, HMZ01b] proved that the first positive eigenvalue of the Dirac operator defined on the compact boundary of a Riemannian Spin manifold (M^n, g) of nonnegative scalar curvature satisfies

$$\lambda_1 \geq \frac{n-1}{2} \inf_M H,$$

where H is the mean curvature of the boundary, assumed to be nonnegative. As an application of the limiting case, they gave an elementary spinorial proof of the famous Alexandrov theorem: *the only compact embedded manifold in* $\mathbb{R}^n$ *of constant mean curvature is the sphere* $\mathbb{S}^{n-1}$ *of dimension* $n-1$.

Recently, $\mathrm{Spin^c}$ geometry became a field of active research with the advent of Seiberg-Witten theory [KM94, Wit94, Sei-Wit94, Fri00]. This theory is based on the fact that every oriented Riemannian compact 4-dimensional manifold has a $\mathrm{Spin^c}$ structure. Applications of the Seiberg-Witten theory to 4-dimensional geometry and topology are already notorious: several theorems arising from Donaldson theory found an elementary proof [Don96]. C. LeBrun [LeB95, LeB96] obtained topological restrictions on 4-dimensional Einstein manifolds and with M.J. Gursky [GL98], they calculated the Yamabe invariant for some 4-dimensional manifolds like the complex projective space $\mathbb{C}P^2$.

From an intrinsic point of view, Spin, almost complex, complex, Kähler, Sasaki and some classes of CR manifolds have a canonical $\mathrm{Spin^c}$ structure. For example, using $\mathrm{Spin^c}$ structures, A. Moroianu [Moro99] proved the Lichnerowicz conjecture on Kähler Spin manifolds which are limiting manifolds for the Kirchberg inequality in even complex dimension [Kir86].

In 2006, O. Hijazi, S. Montiel and F. Urbano [HMU06] constructed on Kähler-Einstein manifolds with nonnegative scalar curvature, $\mathrm{Spin^c}$ structures carrying *Kählerian Killing spinors*. The restriction of these spinors to minimal Lagrangian submanifolds provides topological and geometric restrictions on these submanifolds.

Hence, the restriction of $\mathrm{Spin^c}$ spinors is an effective tool to study the geometry and the topology of submanifolds. Moreover, from the extrinsic point of view, it seems that it is more natural to work with $\mathrm{Spin^c}$ structures rather than Spin structures, which are by now very classic. In this mini-course, we will examine $\mathrm{Spin^c}$ structures on hypersurfaces, to prove a Lawson type correspondence for constant mean curvature surfaces in some 3-dimensional special geometries.

2 Algebraic facts

The aim of this section is to present some algebraic ingredients lying at the heart of $\mathrm{Spin^c}$ geometry. We refer to [LM89, Hij01, BHMM].

We denote by Cl_n the real Clifford algebra. It is the unitary algebra generated by elements $v, w \in \mathbb{R}^n$ such that

$$v \cdot w + w \cdot v = -2 < v, w >_{\mathbb{R}^n},$$

where $< .,. >_{\mathbb{R}^n}$ is the canonical scalar product of $\mathbb{R}^n$ and "$\cdot$" denotes the binary operator. If $\{e_1, \dots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$, then

$$\{1, e_{i_1} \cdot \dots \cdot e_{i_k}, 1 \leq i_1 < \dots < i_k \leq n, 0 \leq k \leq n\}$$

is a basis of Cl_n ($\dim_{\mathbb{R}} \mathrm{Cl}_n = 2^n$). The complex Clifford algebra $\mathbb{C}l_n$ is the complexification of the real one, i.e.,

$$\mathbb{C}l_n = \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}.$$

Examples 2.1 *A basis of* Cl_1 *is given by* $\{1, e_1\}$ *with* $e_1^2 = -1$*, then* $e_1 = i$ *and* $\mathrm{Cl}_1 = \mathbb{C}$*. Moreover,* $\mathbb{C}l_1 = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}$*. A basis of* Cl_2 *is given by* $\{1, e_1, e_2, e_1 \cdot e_2\}$ *with* $e_1^2 = e_2^2 = (e_1 \cdot e_2)^2 = -1$ *and* $e_1 \cdot e_2 = -e_2 \cdot e_1$*. One has* $\mathrm{Cl}_2 = \mathbb{H}$ *and* $\mathbb{C}l_2 = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}(2)$*. Here* $\mathbb{H}$ *denotes the Quaternion and* $\mathbb{C}(2)$ *the set of complex matrix of order* 2*.*

Let $\omega_{\mathbb{C}} := i^{[\frac{n+1}{2}]} e_1 \cdot e_2 \cdot \dots \cdot e_n$ be the complex volume element.

Theorem 2.1 *A complex representation of* $\mathbb{C}l_n$ *is irreducible if and only if it is of complex dimension* $2^{[\frac{n}{2}]}$*, where* $[\frac{n}{2}]$ *denotes the integer part of* $\frac{n}{2}$*. More precisely, if* $n = 2m$ *is even,* $\mathbb{C}l_n$ *has an unique irreducible representation of complex dimension* 2^m*:*

$$\gamma_{2m} : \mathbb{C}l_{2m} \xrightarrow{\simeq} \mathrm{End}_{\mathbb{C}}(\Sigma_{2m} \simeq \mathbb{C}^{2^m}).$$

Moreover, $(\gamma_{2m}(\omega_{\mathbb{C}}))^2 = \mathrm{Id}$*. If* $n = 2m+1$ *is odd,* $\mathbb{C}l_n$ *has two inequivalent irreducible representations both of complex dimension* 2^m*:*

$$\gamma_{2m+1} \text{ and } \gamma'_{2m+1} : \mathbb{C}l_{2m+1} \longrightarrow \mathrm{End}_{\mathbb{C}}(\Sigma_{2m}).$$

Moreover, $\gamma_{2m+1}(\omega_{\mathbb{C}}) = \mathrm{Id}$ *and* $\gamma'_{2m+1}(\omega_{\mathbb{C}}) = -\mathrm{Id}$*.*

We denote by $\mathbb{C}l_n^0$ the even part of $\mathbb{C}l_n$. It is the subalgebra of $\mathbb{C}l_n$ generated by

$$\{1, e_{i_1} \cdot \dots \cdot e_{i_k},\ 1 \leq i_1 < \dots < i_k \leq n,\ 0 \leq k \leq n,\ k \text{ is even}\}.$$

Lemma 2.2 *For every* $n \in \mathbb{N}^*$*, we have* $\mathbb{C}l_n \simeq \mathbb{C}l_{n+1}^0$*.*

Proof. Let $\{e_1, e_2, \dots, e_n, \nu\}$ be an orthonormal basis of $\mathbb{R}^{n+1}$ such that $\{e_1, e_2, \dots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$. The following map

$$\begin{array}{ccc} \mathbb{C}l_n & \longrightarrow & \mathbb{C}l_{n+1}^0 \\ e_{i_1} \cdot \dots \cdot e_{i_k} & \longmapsto & \begin{cases} e_{i_1} \cdot \dots \cdot e_{i_k} & \text{if } k \text{ is even,} \\ e_{i_1} \cdot \dots \cdot e_{i_k} \cdot \nu & \text{if } k \text{ is odd,} \end{cases} \end{array}$$

is an isomorphism.

Proposition 2.3 *For every $m \in \mathbb{N}^*$, we have*

$$\gamma_{2m+1}(\mathbb{C}l^0_{2m+1}) = \mathrm{Aut}_{\mathbb{C}}(\Sigma_{2m}),$$

$$\gamma_{2m}(\mathbb{C}l^0_{2m}) = \mathrm{Aut}_{\mathbb{C}}(\Sigma^+_{2m}) \oplus \mathrm{Aut}_{\mathbb{C}}(\Sigma^-_{2m}),$$

where $\Sigma_{2m} = \Sigma^+_{2m} \oplus \Sigma^-_{2m}$ and $\Sigma^\pm_{2m} = \{\sigma \in \Sigma_{2m}, \gamma_{2m}(\omega_{\mathbb{C}})\sigma = \pm\sigma\}$.

Proof. For $n = 2m+1$, the map $\gamma_{2m+1|_{\mathbb{C}l^0_{2m+1}}} : \mathbb{C}l^0_{2m+1} \longrightarrow \mathrm{End}_{\mathbb{C}}(\Sigma_{2m})$ defines a complex representation of $\mathbb{C}l^0_{2m+1} \simeq \mathbb{C}l_{2m}$ of complex dimension 2^m. But γ_{2m} is the unique irreducible representation of $\mathbb{C}l_{2m}$ of complex dimension 2^m, then $\gamma_{2m} \simeq \gamma_{2m+1|_{\mathbb{C}l^0_{2m+1}}}$. Thus, $\gamma_{2m+1}(\mathbb{C}l^0_{2m+1}) = \mathrm{End}_{\mathbb{C}}(\Sigma_{2m})$. Moreover, for all $u \in \mathbb{C}l^0_{2m+1}$, $\gamma_{2m+1}(u)$ is an isomorphism. In fact, for $u = e_1 \cdot \ldots \cdot e_k$, k even, we set $v = e_k \cdot \ldots \cdot e_1$ and we can check that $\gamma_{2m+1}(u) \circ \gamma_{2m+1}(v) = \mathrm{Id}$.

3 Spin^c structures and the Dirac operator

In this section, we define $\mathrm{Spin^c}$ structures on a Riemannian manifold (M^n, g) according to S. Montiel [Mon05]. A $\mathrm{Spin^c}$ structure on M is needed to define globally a complex vector bundle ΣM *called the spinor bundle*, such that at every point $x \in M$, the fiber is given by $\Sigma_x M = \Sigma_n = \mathbb{C}^{2^{[\frac{n}{2}]}}$. On sections of the spinor bundle ΣM, we then define the Dirac operator and we give its basic properties.

Definition 3.1 *Let (M^n, g) be an oriented compact Riemannian manifold. A Dirac bundle on (M^n, g) is a complex vector bundle ΣM of rank l endowed with*
• *A Hermitian metric $< .,. >$ and a connection ∇ which parallelizes the metric, i.e.,*

$$X(<\psi, \varphi>) = <\nabla_X \psi, \varphi> + <\psi, \nabla_X \varphi>,$$

for all $X \in \Gamma(TM)$ and $\psi, \varphi \in \Gamma(\Sigma M)$.
• *A $C^\infty(M)$-linear map $\gamma : TM \longrightarrow \mathrm{End}(\Sigma M)$ satisfying*

$$<\gamma(X)\psi, \varphi> + <\psi, \gamma(X)\varphi> = 0, \tag{1}$$

$$\nabla_X(\gamma(Y)\psi) = \gamma(\nabla_X Y)\psi + \gamma(Y)\nabla_X \psi, \tag{2}$$

$$\gamma(X)\gamma(Y) + \gamma(Y)\gamma(X) = -2g(X,Y), \tag{3}$$

for all $X, Y \in \Gamma(TM)$, $\psi, \varphi \in \Gamma(\Sigma M)$ and where the second connection on Equation (2) is the Levi-Civita connection on M.

Why do we need the existence of such a map γ ?
Why do we ask γ to satisfy these three conditions ?

On a Riemannian manifold (M^n, g), giving a complex vector bundle $(\Sigma M, \nabla, \langle ., . \rangle)$, we have not any nontrivial natural first order operator acting on $\Gamma(\Sigma M)$. However, we have a second order operator: the Laplacian $\triangle$. When γ exists, we can define a natural first order operator D by

$$\begin{array}{rcl} D : \Gamma(\Sigma M) & \longrightarrow & \Gamma(\Sigma M) \\ \psi & \longmapsto & D\psi = \sum_{j=1}^{n} \gamma(e_j)\nabla_{e_j}\psi, \end{array}$$

where $\{e_j\}_{j=1,...,n}$ is a local orthonormal basis tangent to M. Conditions (1), (2) and (3) are needed to get some properties of D.

Proposition 3.2 *Let $(\Sigma M, \nabla, \langle ., . \rangle, \gamma)$ be a Dirac bundle over a compact manifold M. Then,*

1. *If M is without boundary, D is formally self-adjoint with respect to the L^2-scalar product $(., .) := \int_M < ., . > v_g$, where v_g is the volume element of M (***Conditions (1) and (2)***).*
2. *D^2 and $\triangle$ have the same principal symbol (***Conditions (2) and (3)***).*

Proof. We choose normal coordinates at $x \in M$, i.e., $(\nabla_{e_j} e_k)_x = 0$, $1 \leq k, j \leq n$.

1. For any $\psi, \varphi \in \Gamma(\Sigma M)$, we compute

$$\begin{aligned} < D\psi, \varphi > &= < \sum_{j=1}^{n} \gamma(e_j)\nabla_{e_j}\psi, \varphi > \\ &\overset{(1)}{=} -\sum_{j=1}^{n} < \nabla_{e_j}\psi, \gamma(e_j)\varphi > \\ &= -\sum_{j=1}^{n} [e_j < \psi, \gamma(e_j)\varphi > - < \psi, \nabla_{e_j}(\gamma(e_j)\varphi) >] \\ &\overset{(2)}{=} -\sum_{j=1}^{n} e_j < \psi, \gamma(e_j)\varphi > + < \psi, D\varphi > \\ &= -\mathrm{div}X_1 - i\,\mathrm{div}X_2 + < \psi, D\varphi >, \end{aligned}$$

   where $X_1, X_2 \in \Gamma(TM)$ are defined by

$$g(X_1, Y) + i\, g(X_2, Y) = < \psi, \gamma(Y)\varphi > .$$

   Finally, integrating over M, we get the desired result.

2. For every $\psi \in \Gamma(\Sigma M)$, we have

$$D^2\psi = \Big(\sum_{k=1}^{n} \gamma(e_k)\nabla_{e_k}\Big)\Big(\sum_{j=1}^{n} \gamma(e_j)\nabla_{e_j}\psi\Big)$$

$$\stackrel{(2)}{=} \sum_{k,j=1}^{n} \gamma(e_k)\gamma(e_j)\nabla_{e_k}\nabla_{e_j}\psi$$

$$\stackrel{(3)}{=} -\sum_{j=1}^{n} \nabla_{e_j}\nabla_{e_j}\psi + \sum_{k,j=1;k\neq j}^{n} \gamma(e_k)\gamma(e_j)\nabla_{e_k}\nabla_{e_j}\psi$$

$$= -\sum_{j=1}^{n} \nabla_{e_j}\nabla_{e_j}\psi + \sum_{k<j}^{n} \gamma(e_k)\gamma(e_j)(\nabla_{e_k}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_k}\psi)$$

$$= \triangle\psi + \frac{1}{2}\sum_{i,j=1}^{n} \gamma(e_i)\gamma(e_j)\,\mathcal{R}_{e_i,e_j}\psi,$$

where $\mathcal{R}$ is the tensor curvature associated with the connection ∇ on ΣM.

The condition (3) is the same defining the Clifford algebra on (TM, g). Then, the map γ can be extented to the Clifford bundle $\mathbb{C}l(TM)$: it is the vector bundle over M whose fibers at every $x \in M$ are $\mathbb{C}l(T_xM) \simeq \mathbb{C}l_n$. The extension of γ will also be denoted by γ:

$$\begin{array}{rcl} \gamma : \mathbb{C}l(TM) & \longrightarrow & \mathrm{End}(\Sigma M) \\ X_1 \cdot \ldots \cdot X_k & \longmapsto & \gamma(X_1 \cdot \ldots \cdot X_k) = \gamma(X_1) \circ \ldots \circ \gamma(X_k). \end{array}$$

Hence, at every point $x \in M$, $\gamma : \mathbb{C}l(T_xM) \simeq \mathbb{C}l_n \longrightarrow \mathrm{End}(\Sigma_x M) \simeq \mathrm{End}(\mathbb{C}^l)$ is a representation of $\mathbb{C}l_n$ of complex dimension l.

Definition 3.3 *A* Spin^c *structure on* (M^n, g) *is a Dirac bundle* $(\Sigma M, \nabla, <.,.>, \gamma)$ *of rank* $l = 2^{[\frac{n}{2}]}$. *In other terms, it is a Dirac bundle supplying at every point* $x \in M$ *an irreducible representation of* $\mathbb{C}l_n$. *In this case,* γ *is called the Clifford multiplication,* ΣM *the spinor bundle, a section* $\psi \in \Gamma(\Sigma M)$ *is called a spinor field and* D *the associated Dirac operator.*

Proposition 3.4 *Let* (M^n, g) *be a Riemannian* Spin^c *manifold. Then, the determinant line bundle* $\det \Sigma M$ *has a root of index* $2^{[\frac{n}{2}]-1}$, *i.e., there exists a complex line bundle* $\mathcal{L}$ *on* M *such that*

$$\underbrace{\mathcal{L} \otimes \cdots \otimes \mathcal{L}}_{2^{[\frac{n}{2}]-1}\text{ times}} = \mathcal{L}^{2^{[\frac{n}{2}]-1}} = \det \Sigma M.$$

The complex line bundle $\mathcal{L}$ *is called the auxiliary line bundle associated with the* Spin^c *structure.*

Proof. ($n = 2m + 1$). We denote by $\varphi_{ij} : U_i \cap U_j \longrightarrow \mathrm{Aut}(\Sigma_{2m})$ the transition functions of the spinor bundle ΣM. We recall that at every $x \in M$, γ is given by γ_{2m+1} and by Proposition 2.3, we have $\gamma_{2m+1}(\mathbb{C}l^0_{2m+1}) = \mathrm{Aut}(\Sigma_{2m})$. Then, we define

$$\psi_{ij} : U_i \cap U_j \xrightarrow{\varphi_{ij}} \mathrm{Aut}(\Sigma_{2m}) \xrightarrow{\gamma^{-1}_{2m+1}} \mathbb{C}l^0_{2m+1},$$

i.e., $\varphi_{ij} = \gamma_{2m+1} \circ \psi_{ij}$. Moreover, for all $X \in \Gamma(TM)$, we know that $\gamma(X)^2 = -|X|^2$, then $(\det \gamma(X))^2 = |X|^{2^{[\frac{n}{2}]+1}}$. Let us define

$$\begin{array}{rcl} \mathrm{sq} : \mathbb{C}l^0_{2m+1} & \longrightarrow & \mathbb{C}^* \\ \lambda v_1 \cdot \ldots \cdot v_{2k} & \longmapsto & \lambda^2 |v_1|^2 \ldots |v_{2k}|^2. \end{array}$$

It is easy to check that at each point $x \in M$,

$$(\det \varphi_{ij}(x))^2 = (\det \gamma_{2m+1}(\psi_{ij}(x)))^2 = [\mathrm{sq}(\psi_{ij}(x))]^{2^{[\frac{n}{2}]}}.$$

Hence the line bundle $\mathcal{L}$ whose transition functions are $l_{ij} = \mathrm{sq}(\psi_{ij})$ satisfies $\mathcal{L}^{2^{[\frac{n}{2}]-1}} = \det \Sigma M$.

4 Examples and remarks

Let (M^n, g) be a Riemannian manifold. Is it always possible to find a Dirac bundle ΣM with rank $2^{[\frac{n}{2}]}$, yielding an irreducible representation of $\mathbb{C}l_n \simeq \mathbb{C}l(T_xM)$ on the vector space $\Sigma_x M \simeq \Sigma_n$ at every $x \in M$?

Proposition 4.1 *([Mon05, Nak11b, Fri00, Moro97]) Every Kähler manifold has a canonical* Spin^c *structure carrying parallel spinors, i.e., there exists a spinor field* ψ *satisfying* $\nabla \psi = 0$.

Proof. Let $(M^{n=2m}, g, J)$ be a Kähler manifold of complex dimension m. The endomorphism $J : TM \longrightarrow TM$ satisfying $J^2 = -1$, can be extented to the complexified tangent bundle $T^cM = TM \otimes_{\mathbb{R}} \mathbb{C}$,

$$J : T^cM \longrightarrow T^cM.$$

It satisfies also $J^2 = -1$. Then,

$$T^cM = T_{1,0}M \oplus T_{0,1}M,$$

where $T_{1,0}M$ (resp. $T_{0,1}M$) is the eigensubbundle of T^cM corresponding to the eigenvalue i (resp. $-i$). The bundle of complex r-forms of type $(1,0)$ is defined by

$$\Lambda^{r,0}M := \Lambda^r(T^*_{1,0}M).$$

For example, if $r = m$, the complex line bundle $K_M := \Lambda^{m,0}M$ is called the canonical bundle of the Kähler manifold. We set $\Sigma M = \Lambda^{*,0}M := \oplus_{r=0}^m \Lambda^r(T^*_{1,0}M)$. It is a complex bundle over M of rank $2^m = 2^{[\frac{n}{2}]}$. We define on $T_{1,0}M$ a Hermitian metric given by

$$< Z, W >= g_{\mathbb{C}}(Z, \overline{W}),$$

for any $Z, W \in T_{1,0}M$. Here $g_{\mathbb{C}}$ denotes the complexification of g. The extension of the Levi-Civita connection to $T_{1,0}M$ and the Hermitian metric $< .,. >$ are compatible. We define γ by

$$\begin{array}{rcl} \gamma : TM & \longrightarrow & \mathrm{End}(\Lambda^{*,0}M) \\ X & \longrightarrow & \gamma(X)\omega = \frac{1}{\sqrt{2}}(X - iJX)^\flat \wedge \omega - \sqrt{2}(X \lrcorner \omega), \end{array}$$

where $(X - iJX)^\flat$ is the complex vector $X - iJX$ viewed as a complex 1-form. Using the properties of $\wedge$ and $\lrcorner$, we can prove that γ satisfies Conditions (1), (2) and (3). Hence, M has a $\mathrm{Spin^c}$ structure carrying parallel spinors (the complex constant functions). Let us find the auxiliary line bundle associated with this $\mathrm{Spin^c}$ structure. We have $\mathcal{L} = \det(\Lambda^{*,0}M)^{2^{1-m}}$. Then, the first Chern class c_1 of $\mathcal{L}$ is given by

$$\begin{aligned} c_1(\mathcal{L}) &= 2^{1-m} c_1(\det\Lambda^{*,0}M) = 2^{1-m} c_1(\Lambda^{*,0}M) \\ &= 2^{1-m}\sum_r^m c_1(\Lambda^{r,0}M) = 2^{1-m}\sum_r^m C_m^r\, c_1(T^*_{1,0}M) \\ &= c_1(T^*_{1,0}M) = c_1(\Lambda^m T^*_{1,0}M) = c_1(K_M). \end{aligned}$$

So, $\mathcal{L} = K_M$.

Remarks 4.2 *1. Let (M^n, g) be a* $\mathrm{Spin^c}$ *manifold. The auxiliary line bundle associated with the* $\mathrm{Spin^c}$ *structure satisfies [Mon05, Fri00]*

$$\omega_2(M) = [c_1(\mathcal{L})]_2,$$

where $\omega_2(M) \in \mathbb{H}^2(M, \mathbb{Z}_2)$ is the second Steifel-Whitney class of the manifold M and $c_1(\mathcal{L})$ the first Chern class of the auxiliary line bundle $\mathcal{L}$. Conversely, the existence of a complex line bundle $\mathcal{L}$ over M satisfying $\omega_2(M) = [c_1(\mathcal{L})]_2$ defines a $\mathrm{Spin^c}$ *structure on M whose auxiliary line bundle is $\mathcal{L}$ [Mon05, Fri00].*

2. *When the auxiliary line bundle $\mathcal{L}$ is a square, i.e., there exists a complex line bundle $\mathcal{V}$ on M, such that $\mathcal{V} \otimes \mathcal{V} = \mathcal{L}$, then $\omega_2(M) = 0$ and the* $\mathrm{Spin^c}$ *structure is called a* Spin *structure.*

3. *Let (M^n, g) be a Riemannian* $\mathrm{Spin^c}$ *manifold. For every complex line bundle L endowed with a connection ∇^L and a Hermitian metric $< .,. >_L$,*

$$\Sigma' M = \Sigma M \otimes L,$$

defines another $\mathrm{Spin^c}$ *structure on M. In this case,*

$$\gamma'(X)(\psi \otimes l) = \gamma(X)\psi \otimes l, \tag{4}$$

$$\nabla' = \nabla \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^L, \tag{5}$$

$$< .,. >' = < .,. >< .,. >_L, \tag{6}$$

$$\mathcal{L}' = \mathcal{L} \otimes L^2. \tag{7}$$

Conversely, for any two $\mathrm{Spin^c}$ *structures on M, there exists a complex line bundle L endowed with a connection ∇^L and a Hermitian metric $< .,. >_L$ such that $\Sigma' M = \Sigma M \otimes L$ [Mon05]. In addition, γ and γ', ∇ and ∇', $< .,. >$ and $< .,. >'$, $\mathcal{L}$ and $\mathcal{L}'$ are related by (4), (5), (6)and (7). When we have a* Spin *structure, the auxiliary line bundle $\mathcal{L}$ can be chosen to be trivial. In fact, since M is a* Spin *manifold, we have $\mathcal{L} = \mathcal{V}^2$ where $\mathcal{V}$ is a complex line bundle over M. Let $\Sigma' M = \Sigma M \otimes \mathcal{V}^{-1}$. This* Spin *structure has a trivial auxiliary line bundle.*

Corollary 4.3 *The complex projective space $\mathbb{C}P^m$ is a* Spin *manifold if m is odd and if m is even it is not a* Spin *manifold.*

Proof. Since $\mathbb{C}P^m$ is a Kähler manifold, it carries a canonical Spin^c structure whose auxiliary line bundle $\mathcal{L}$ is $K_{\mathbb{C}P^m}$ (see Proposition 4.1). It is known that the index of $\mathbb{C}P^m$ is $m+1$ [HMU06]. It is the greatest number such that there exists a complex line bundle $\mathcal{V}$ over $\mathbb{C}P^m$ satisfying $\mathcal{V}^{m+1} = K_{\mathbb{C}P^m}$. Moreover, the line bundle $\mathcal{V}$ is the tautological bundle of $\mathbb{C}P^m$. On the other hand, the spinor bundle of any other Spin^c structure on $\mathbb{C}P^m$ can be written as [Fri00, HMU06]:

$$\Sigma M = \Lambda^{*,0}\mathbb{C}P^m \otimes \mathcal{V}^{\frac{q-m-1}{2}},$$

where $q \in \mathbb{Z}$ and it satisfies $q-m-1 \in 2\mathbb{Z}$. We recall that the auxiliary line bundle of this new Spin^c structure is given by $K_{\mathbb{C}P^m} \otimes \mathcal{V}^{q-m-1} = \mathcal{V}^q$. Then, if m is odd, q could be 0. In this case, $\mathcal{V}^{q=0}$ is trivial and the Spin^c structure for $q=0$ is a Spin structure. But for m even, q is always odd. Hence $\mathcal{V}^q$ cannot be a square and then $\mathbb{C}P^m$ is not a Spin manifold.

5 The Schrödinger-Lichnerowicz formula

An important tool when examining the Dirac operator is the Schrödinger-Lichnerowicz formula. It relates the square of the Dirac operator to some geometric data, like the scalar curvature.

Proposition 5.1 *(***The Levi-Civita Spin^c connection***) Let (M^n, g) be a Riemannian* Spin^c *manifold. For every $X \in \Gamma(TM)$ and a local spinor field $\psi : U \subset M \longrightarrow \Sigma M$, we have*

$$\nabla_X \psi = \frac{1}{4}\sum_{j=1}^{n} \gamma(e_j)\gamma(\nabla_X e_j)\psi + \frac{i}{2}\alpha(X)\psi,$$

where $i\alpha : TU \longrightarrow i\mathbb{R}$ is an imaginary local 1-form. It is the local expression of the connection on the auxiliary line bundle $\mathcal{L}$.

Proof. We denote by $\nabla^0_X\psi = \frac{1}{4}\sum_{j=1}^{n}\gamma(e_j)\gamma(\nabla_X e_j)\psi$. We can check (exercice) that ∇^0 satisfies the same condition (2) as ∇, i.e., for all $X, Y \in \Gamma(TM)$

$$\nabla^0_X(\gamma(Y)\psi) = \gamma(\nabla_X Y)\psi + \gamma(Y)\nabla^0_X\psi.$$

Then $\nabla'_X = \nabla_X - \nabla^0_X$ satisfies $\nabla'_X(\gamma(Y)\psi) = \gamma(Y)\nabla'_X\psi$. Hence, ∇' commutes with $\gamma(T_xM)$ for all $x \in U$ and so with $\gamma(\mathbb{C}l(T_xM))$ for all $x \in U$. But $\nabla'_X \in \mathrm{End}(\Sigma_x M)$ and since it commutes with $\mathrm{End}(\Sigma_x M) \subset \gamma(\mathbb{C}l(T_xM))$, we have that ∇'_X is in the center of $\mathrm{End}(\Sigma_x M)$ which is trivial. Then, there exists on U a complex 1-form β such that $\nabla'_X = \beta(X)\mathrm{Id}$. Now, we endow $\mathcal{L} = (\det \Sigma M)^{2^{1-[\frac{n}{2}]}}$ with the connection induced from ∇ on ΣM. Locally, we have

$$\begin{aligned} i\alpha(X) &= 2^{1-[\frac{n}{2}]}\nabla^{\det \Sigma M}_X = 2^{1-[\frac{n}{2}]}\mathrm{tr}(\nabla_X) \\ &= 2^{1-[\frac{n}{2}]}\mathrm{tr}(\nabla^0_X + \beta(X)\mathrm{Id}) = 2^{1-[\frac{n}{2}]}2^{[\frac{n}{2}]}\beta(X) = 2\beta(X), \end{aligned}$$

because ∇^0_X is traceless.

Theorem 5.2 *(***The Schrödinger-Lichnerowicz formula***) Let (M^n, g) be a Riemannian* $\mathrm{Spin}^{\mathrm{c}}$ *manifold. Then,*

$$D^2 = \triangle + \frac{1}{4}\mathrm{Scal} + \frac{i}{2}\gamma(\Omega),$$

where $i\Omega = id\alpha$ is the curvature 2-form on $\mathcal{L}$ (it is an imaginary global 2-form) and $\gamma(\Omega)$ is the extension of γ to forms given by $\gamma(X \wedge Y) = \gamma(X)\gamma(Y)$ for all $X, Y \in \Gamma(TM)$.

Proof. From Proposition 5.1, we can compute the curvature $\mathcal{R}$ on ΣM associated with ∇ and we get, for all $X, Y \in \Gamma(TM)$,

$$\mathcal{R}(X,Y) = \mathcal{R}^0(X,Y) + \frac{i}{2}\Omega(X,Y),$$

where $\mathcal{R}^0(X,Y) = \frac{1}{4}\sum_{i,j=1}^n g(R(X,Y)e_i, e_j)\gamma(e_i)\gamma(e_j)$ and R is the Riemannian tensor. Moreover from Proposition 3.2, we have

$$\begin{aligned} D^2 &= \triangle + \frac{1}{2}\sum_{i,j=1}^n \gamma(e_i)\gamma(e_j)\mathcal{R}(e_i, e_j) \\ &= \triangle + \frac{1}{2}\sum_{i,j=1}^n \gamma(e_i)\gamma(e_j)\mathcal{R}^0(e_i, e_j) + \frac{i}{4}\sum_{i,j=1}^n \Omega(e_i, e_j)\gamma(e_i)\gamma(e_j). \end{aligned}$$

Denoting by $R_{ijkl} = g(R(e_i, e_j)e_k, e_l)$, we have for all $\psi \in \Gamma(\Sigma M)$,

$$\begin{aligned} \frac{1}{2}\sum_{i,j=1}^n \gamma(e_i)\gamma(e_j)\mathcal{R}^0(e_i, e_j)\psi &= \frac{1}{8}\sum_{i,j,k,l=1}^n R_{ijkl}\gamma(e_i)\gamma(e_j)\gamma(e_k)\gamma(e_l)\psi \\ &\overset{(3)}{=} \frac{1}{8}\sum_{l=1}^n \Big(\frac{1}{3}\sum_{i\neq j\neq k}(R_{ijkl} + R_{jkil} + R_{kijl})\gamma(e_i)\gamma(e_j)\gamma(e_k) \\ &\quad + \sum_{i,j} R_{ijil}\gamma(e_i)\gamma(e_j)\gamma(e_i) \\ &\quad + \sum_{i,j} R_{ijjl}\gamma(e_i)\gamma(e_j)\gamma(e_j)\Big)\gamma(e_l)\psi \\ &\overset{(3)}{=} \frac{1}{8}\Big(\sum_{l,j} -\mathrm{Ric}(e_j, e_l)\gamma(e_l)\gamma(e_j)\psi - \sum_{i,l} -\mathrm{Ric}(e_i, e_l)\gamma(e_i)\gamma(e_l)\psi\Big) \\ &= -\frac{1}{4}\sum_{i,j=1}^n \mathrm{Ric}(e_i, e_j)\gamma(e_i)\gamma(e_j)\psi \\ &\overset{(3)}{=} \frac{1}{4}\mathrm{Scal}\,\psi. \end{aligned}$$

Similary, we have

$$\frac{i}{4}\sum_{k,j=1}^n \Omega(e_k, e_j)\gamma(e_k)\gamma(e_j)\psi = \frac{i}{2}\sum_{k<j}\Omega(e_k, e_j)\gamma(e_k)\gamma(e_j)\psi = \frac{i}{2}\gamma(\Omega)\psi.$$

6 Hypersurfaces of Spin^c manifolds

Now, we move to study $\mathrm{Spin^c}$ structures on hypersurfaces, such as the restriction of the spinor bundle of an ambient manifold and the $\mathrm{Spin^c}$ Gauss formula.

Proposition 6.1 *([Nak11b, Mon05]) Every real oriented hypersurface (M^n, g) of a* $\mathrm{Spin^c}$ *manifold $\mathcal{Z}^{n+1}$ is also a* $\mathrm{Spin^c}$ *manifold.*

Proof. ($n = 2m$) Let $M^n \hookrightarrow \mathcal{Z}$ be an isometric immersion of M into $\mathcal{Z}$. We denote by ν the normal vector field of the immersion and by II the second fundamental form, $II(X) = -\nabla^{\mathcal{Z}}_X \nu$, for all $X \in \Gamma(TM)$. The complex vector bundle $\Sigma M := \Sigma\mathcal{Z}_{|_M}$ is of rank $2^{[\frac{n+1}{2}]} = 2^{[\frac{n}{2}]} = 2^{\frac{n}{2}}$. Now, if we restrict the Levi-Civita $\mathrm{Spin^c}$ connection $\nabla^{\mathcal{Z}}$ to M, we do not obtain a suitable connection because Condition (2) implies

$$\nabla^{\mathcal{Z}}_X(\gamma(Y)\varphi) = \gamma(\nabla^{\mathcal{Z}}_X Y)\varphi + \gamma(Y)\nabla_X\varphi,$$

for all $X, Y \in \Gamma(TM)$, $\varphi \in \Gamma(\Sigma M)$, where γ denotes the Clifford multiplication on $\mathcal{Z}$. Since $\nabla^{\mathcal{Z}}_X Y$ is the Levi-Civita on $\mathcal{Z}$ and not on M, we do not have a suitable connection. We want to define $\gamma^M : TM \longrightarrow \mathrm{End}(\Sigma M)$ such that at every point $x \in M$, $\gamma^M : \mathbb{C}l(T_xM) \longrightarrow \mathrm{End}(\Sigma_x M)$ defines an irreducible representation of $\mathbb{C}l(T_xM) \simeq \mathbb{C}l_n$. We know that, at every $x \in M$, γ is an irreducible representation of $\mathbb{C}l_{2m+1}$ and by Proposition 2.3, $\gamma_{2m+1}(\mathbb{C}l^0_{2m+1}) = \mathrm{End}(\mathbb{C}^{2^{[\frac{n}{2}]}}) = \mathrm{End}(\mathbb{C}^{2^{[\frac{n+1}{2}]}})$. Then,

$$\begin{array}{ccccc}
\mathbb{C}l_{2m} & \overset{\simeq}{\longrightarrow} & \mathbb{C}l^0_{2m+1} & \overset{\gamma_{2m+1}}{\longrightarrow} & \mathrm{End}(\mathbb{C}^{2^{[\frac{n+1}{2}]}}) \\
e_j & \longmapsto & e_j \cdot \nu & \longmapsto & \gamma_{2m+1}(e_j)\gamma_{2m+1}(\nu)
\end{array}$$

defines an irreducible representation of $\mathbb{C}l_{2m}$. So, we set $\gamma^M(X) = \gamma(X)\gamma(\nu)$ and γ^M satisfies Conditions (1) and (3). Now, if ∇^M is a suitable connection on M, then we should have

$$\nabla^M_X \varphi = \frac{1}{4}\sum_{j=1}^{n} \gamma^M(e_j)\gamma^M(\nabla^M_X e_j)\varphi + \frac{i}{2}\alpha^{\mathcal{L}^M}(X)\varphi,$$

for every $X \in \Gamma(TM)$, $\varphi = \psi|_M \in \Gamma(\Sigma M)$ and where $i\alpha^{\mathcal{L}^M}$ is local expression of the connection on the auxiliary line bundle $\mathcal{L}^M$. Moreover, we have the three following facts:

1. For all $X, Y \in \Gamma(TM)$, we have
$$\gamma^M(X)\gamma^M(Y) = \gamma(X)\gamma(\nu)\gamma(Y)\gamma(\nu) = \gamma(X)\gamma(Y).$$
2. The auxiliary line bundle $\mathcal{L}^M$ is given by
$$\mathcal{L}^M = (\det \Sigma M)^{2^{1-[\frac{n}{2}]}} = (\det \Sigma\mathcal{Z}_{|_M})^{2^{1-[\frac{n}{2}]}} = (\det \Sigma\mathcal{Z})^{2^{1-[\frac{n+1}{2}]}}{}_{|_M} = \mathcal{L}^{\mathcal{Z}}{}_{|_M}.$$
Then $i\alpha^{\mathcal{L}^{\mathcal{Z}}}(X) = i\alpha^{\mathcal{L}^M}(X)$ for all $X \in \Gamma(TM)$.

3. The Levi-Civita connections ∇^M and $\nabla^{\mathcal{Z}}$ are related by the Gauss formula $\nabla^{\mathcal{Z}}_X Y = \nabla^M_X Y + II(X,Y)\nu$, for all $X, Y \in \Gamma(TM)$.

Then, we have

$$\begin{aligned}
\nabla^M_X \varphi &= \frac{1}{4}\sum_{j=1}^n \gamma(e_j)\gamma(\nabla^{\mathcal{Z}}_X e_j)\psi|_M - \frac{1}{4}\sum_{j=1}^n II(X,e_j)\gamma(e_j)\gamma(\nu)\psi|_M + \frac{i}{2}\alpha^{\mathcal{L}^{\mathcal{Z}}}(X)\psi|_M \\
&= \nabla^{\mathcal{Z}}_X\psi|_M - \frac{1}{4}\gamma(\nu)\gamma(\nabla^{\mathcal{Z}}_X\nu)\psi|_M - \frac{1}{4}\gamma(II(X))\gamma(\nu)\psi|_M \\
&= \nabla^{\mathcal{Z}}_X\psi|_M + \frac{1}{4}\gamma(\nu)\gamma(II(X))\psi|_M - \frac{1}{4}\gamma(II(X))\gamma(\nu)\psi|_M \\
&\overset{(3)}{=} \nabla^{\mathcal{Z}}_X\psi|_M - \frac{1}{2}\gamma(II(X))\gamma(\nu)\psi|_M \\
&= \nabla^{\mathcal{Z}}_X\psi|_M - \frac{1}{2}\gamma^M(II(X))\varphi. \quad \textbf{(The Spin}^c \textbf{ Gauss formula)}
\end{aligned}$$

Choosing ∇^M to be $\nabla^M_X\varphi = \nabla^{\mathcal{Z}}_X\psi|_M - \frac{1}{2}\gamma^M(II(X))\varphi$, for all $\varphi = \psi|_M \in \Gamma(\Sigma M)$ and $X \in \Gamma(TM)$, we get a suitable connection satisfying Condition (2).

Corollary 6.2 *Let M^n (n even) be a real oriented hypersurface isometrically immersed into a* $\mathrm{Spin^c}$ *manifold $\mathcal{Z}$ of mean curvature $H = \frac{1}{n}\mathrm{tr}II$. Then,*

1. *$D^M\varphi = \frac{n}{2}H\varphi - \gamma(\nu)D^{\mathcal{Z}}\psi|_M - \nabla^{\mathcal{Z}}_\nu\psi|_M$, where $\varphi = \psi|_M$ and D^M (resp. $D^{\mathcal{Z}}$) is the Dirac operator on M (resp. on $\mathcal{Z}$).*

2. *Denoting by $i\Omega^{\mathcal{Z}}$ (resp. $i\Omega^M$) the curvature of the auxiliary line bundle $\mathcal{L}^{\mathcal{Z}}$ (resp. $\mathcal{L}^M$), we have*

$$\gamma(\Omega^{\mathcal{Z}})\psi|_M = \gamma^M(\Omega^M)\varphi - \gamma^M(\nu\lrcorner\Omega^{\mathcal{Z}})\varphi, \qquad (8)$$

Proof. For every $\varphi = \psi|_M$, we have

$$\begin{aligned}
D^M\varphi &= \sum_{j=1}^n \gamma^M(e_j)\nabla^M_{e_j}\varphi \\
&= \sum_{j=1}^n \gamma(e_j)\gamma(\nu)\Big(\nabla^{\mathcal{Z}}_{e_j}\psi|_M - \frac{1}{2}\gamma(II(e_j))\gamma(\nu)\psi|_M\Big) \\
&= -\sum_{j=1}^n \gamma(\nu)\gamma(e_j)\nabla^{\mathcal{Z}}_{e_j}\psi|_M + \frac{1}{2}\sum_{j=1}^n \gamma(e_j)\gamma(II(e_j))\psi|_M \\
&= -\gamma(\nu)D^{\mathcal{Z}}\psi|_M + \gamma(\nu)(\gamma(\nu)\nabla^{\mathcal{Z}}_\nu\psi|_M + \sum_{k,j=1}^n II(e_j,e_k)\gamma(e_j)\gamma(e_k)\psi|_M \\
&\overset{(3)}{=} -\gamma(\nu)D^{\mathcal{Z}}\psi|_M - \nabla^{\mathcal{Z}}_\nu\psi|_M + \frac{n}{2}H\varphi.
\end{aligned}$$

Moreover,

$$\begin{aligned}\Omega^{\mathcal{Z}} &= \sum_{j<k}^{n} \Omega^{\mathcal{Z}}(e_j, e_k) e_j \wedge e_k + \sum_{j=1}^{n} \Omega^{\mathcal{Z}}(e_j, \nu) e_j \wedge \nu \\ &= \sum_{j<k}^{n} \Omega^{M}(e_j, e_k) e_j \wedge e_k - \nu \lrcorner \Omega^{\mathcal{Z}} \wedge \nu.\end{aligned}$$

Finally, $\gamma(\Omega^{\mathcal{Z}})\psi|_M = \gamma^M(\Omega^M)\varphi - \gamma^M(\nu \lrcorner \Omega^{\mathcal{Z}})\varphi$.

7 Geometric applications

A Riemannian manifold is said to be homogeneous if its isometry group acts transitively on it, i.e., for any two points p and q, there exists an isometry that maps p to q. A homogeneous manifold is necessarily complete. It is a classical result of Riemannian geometry that a homogeneous 2-manifold has constant curvature. Consequently, up to homotheties there are only three simply connected homogeneous 2-manifolds: the Euclidean plane $\mathbb{R}^2$, the sphere $\mathbb{S}^2$ and the hyperbolic plane $\mathbb{H}^2$. In dimension 3, the classifcation of simply connected homogeneous manifolds is also well-known but more examples arise. Such a manifold has an isometry group of dimension 3, 4 or 6. When the dimension of the isometry group is 6, then we have a space form ($\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$). When the isometry group has dimension 3, then we have the solvable group Sol_3. The ones with a 4-dimensional isometry group are denoted by $\mathbb{E}(\kappa, \tau)$. All manifolds $\mathbb{E}(\kappa, \tau)$ have the property that there exists a Riemannian fibration

$$\mathbb{E}(\kappa, \tau) \longrightarrow \mathbb{M}^2(\kappa),$$

over the simply connected surface $\mathbb{M}^2(\kappa)$ of curvature κ with bundle curvature τ. The bundle curvature τ measures the defect to be a product. When $\tau = 0$, the fibration is trivial, i.e., $\mathbb{E}(\kappa, \tau)$ is nothing but the product space $\mathbb{M}^2(\kappa) \times \mathbb{R}$. There exist five different kinds of manifolds according to the parameters τ and κ: the product spaces $\mathbb{S}^2(\kappa) \times \mathbb{R}$ and $\mathbb{H}^2(\kappa) \times \mathbb{R}$, Berger spheres, the Heisenberg group Nil_3 and the universal cover of the Lie group $\mathrm{PSL}_2(\mathbb{R})$. Homogeneous 3-manifolds are also related to "Thurston geometries". In fact, all homogeneous manifolds of dimension 3, i.e., $\mathbb{R}^3, \mathbb{H}^3, \mathbb{S}^3, \mathrm{Sol}_3$ and all $\mathbb{E}(\kappa, \tau)$, except Berger spheres, are the eight geometries of Thurston.

Proposition 7.1 *Let (M^2, g) be a simply connected oriented surface isometrically immersed into $\mathbb{H}^2(-1) \times \mathbb{R} = \mathbb{H}^2 \times \mathbb{R}$ of mean curvature H. Then, there exists a* Spin^c *structure on M carrying a spinor field φ of constant norm satisfying $D^M\varphi = H\varphi$. Moreover, in any local orthonormal frame $\{e_1, e_2\}$ of M, the curvature of the auxiliary bundle $\mathcal{L}^M$ associated with this* Spin^c *structure is given by $i\Omega^M(e_1, e_2) = i < \varphi, \frac{\overline{\varphi}}{|\varphi|^2} >$, where $\overline{\varphi} = \varphi_+ - \varphi_-$ is given by the decomposition of $\varphi = \varphi_+ + \varphi_-$ into positive and negative spinors.*

Proof. The manifold $\mathbb{H}^2$ is Kähler, so by Proposition 4.1, it has a Spin^c structure carrying a parallel spinor field. Also, $\mathbb{R}$ is a Spin manifold carrying a parallel spinor field [Hij01]. So, the product $\mathbb{H}^2 \times \mathbb{R}$ is also a Spin^c manifold carrying a parallel spinor field ψ [Moro97, Nak11b]. We endow M with the restricted Spin^c structure and let $\varphi = \psi|_M$. Since ψ is a parallel spinor, we have for all $X \in T(\mathbb{H}^2 \times \mathbb{R})$,

$$X(|\psi|^2) =< \nabla_X^{\mathbb{H}^2\times\mathbb{R}}\psi, \psi > + < \psi, \nabla_X^{\mathbb{H}^2\times\mathbb{R}}\psi >= 0,$$

i.e., ψ is of constant norm, and so φ is also of constant norm. By Corollary 6.2, we have

$$D^M\varphi = H\varphi - \gamma(\nu)D^{\mathbb{H}^2\times\mathbb{R}}\psi|_M - \nabla_\nu^{\mathbb{H}^2\times\mathbb{R}}\psi|_M = H\varphi.$$

It remains to show that $i\Omega^M(e_1, e_2) = i < \varphi, \frac{\overline{\varphi}}{|\varphi|^2} >$. By the Schrödinger-Lichnerowicz formula (see Theorem 5.2), we have $\gamma(\Omega^{\mathbb{H}^2\times\mathbb{R}})\psi = -i\psi$. Moreover, using that, in even dimension, $\gamma^M(\omega^{\mathbb{C}})\varphi = \overline{\varphi}$, we get

$$\begin{aligned}\gamma^M(\Omega^M)\varphi &= \Omega^M(e_1, e_2)\gamma^M(e_1)\gamma^M(e_2)\varphi \\ &= -i\Omega^M(e_1, e_2)\gamma^M(\omega^{\mathbb{C}})\varphi \\ &= -i\Omega^M(e_1, e_2)\overline{\varphi}.\end{aligned}$$

Now, taking the scalar product of (8) with $\overline{\varphi}$, we have

$$-i < \varphi, \overline{\varphi} >= -i\Omega^M(e_1, e_2)|\varphi|^2 - < \gamma^M(\nu \lrcorner \Omega^{\mathbb{H}^2\times\mathbb{R}})\varphi, \overline{\varphi} > .$$

Finally, we can check (exercice) that $< \gamma^M(\nu \lrcorner \Omega^{\mathbb{H}^2\times\mathbb{R}})\varphi, \overline{\varphi} >= 0$, and we obtain $\Omega^M(e_1, e_2) =< \varphi, \frac{\overline{\varphi}}{|\varphi|^2} >$.

Remarks 7.2 *1. The converse of Proposition 7.1 is also true, i.e., having a simply connected* Spin^c *surface* (M^2, g) *carrying a spinor field* φ *of constant norm, satisfying the Dirac equation* $D^M\varphi = H\varphi$ *such that* $\Omega^M(e_1, e_2) =< \varphi, \frac{\overline{\varphi}}{|\varphi|^2} >$, *we can immerse* M *into* $\mathbb{H}^2 \times \mathbb{R}$. *In fact, B. Daniel [Dan09, Dan07] proved that to immerse a simply connected surface* M *into* $\mathbb{H}^2 \times \mathbb{R}$, *we should have a symmetric endomorphism* E *on* M, *a vector field* T *and a real function* f *satisfying*

$$K = \det E - f^2 \ \textbf{(Gauss equation)}, \tag{9}$$

$$\|T\|^2 + f^2 = 1, \tag{10}$$

$$d^\nabla E(X, Y) = -f(g(Y, T)X - g(X, T)Y), \ \textbf{(Codazzi equation)} \tag{11}$$

$$\nabla_X T = fEX, \tag{12}$$

$$X(f) = -g(EX, T), \tag{13}$$

where K *is the Gauss curvature of* M. *Having a* Spin^c *structure on* M *carrying a spinor field* φ *of constant norm satisfying* $D^M\varphi = H\varphi$ *and* $i\Omega^M(e_1, e_2) = i < \varphi, \frac{\overline{\varphi}}{|\varphi|^2} >$, *there exists a natural choice [Nak11b, NR11] of* f, T *and* E *using* *the spinor φ and such that Equations (9), (10), (11), (12) and (13) are satisfied. In fact, we have $f =< \varphi, \frac{\overline{\varphi}}{|\varphi|^2} >$, T is defined by*

$$g(T, e_1) =< i\gamma^M(e_2)\varphi, \frac{\overline{\varphi}}{|\varphi|^2} >, \text{ and } g(T, e_2) = - < i\gamma^M(e_1)\varphi, \frac{\overline{\varphi}}{|\varphi|^2} >,$$

and $E = \ell^\varphi$, where ℓ^φ is the energy-momentum tensor associated with φ. It is a symmetric 2-tensor defined by

$$\ell^\varphi(X, Y) = \mathrm{Re} < \gamma^M(X)\nabla^M_Y\varphi + \gamma^M(Y)\nabla^M_X\varphi, \frac{\varphi}{|\varphi|^2} >,$$

for all $X, Y \in \Gamma(TM)$. This tensor has been studied by many authors (see [Hij95, BGM05, Ha-Na10, Nak11a]).

2. *We can also characterise simply connected surfaces in $\mathbb{S}^2 \times \mathbb{R}$ [Ha-Na10, NR11], Berger Spheres, the Heinsenberg group Nil_3 or the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ [NR11]. In fact, $\mathbb{E}(\kappa, \tau)$, with $\tau = 0$, are Spin^c manifolds carrying a parallel spinor field. When $\tau \neq 0$, $\mathbb{E}(\kappa, \tau)$ are Spin^c manifolds carrying a Killing spinor field of Killing constant $\frac{\tau}{2}$ [NR11], i.e., a spinor field ψ satisfying*

$$\nabla^{\mathbb{E}(\kappa,\tau)}_X\psi = \frac{\tau}{2}\gamma(X)\psi,$$

for all $X \in \Gamma(T\mathbb{E}(\kappa, \tau))$. As for $\mathbb{H}^2 \times \mathbb{R}$, the restriction to M of the Killing spinor field ψ defines a spinor field φ on M of constant norm and satisfying $D^M\varphi = H\varphi - i\tau\overline{\varphi}$. The curvature of the auxiliary line bundle is given by $i\Omega^M(e_1, e_2) = -i(\kappa - 4\tau^2) < \varphi, \frac{\overline{\varphi}}{|\varphi|^2} >$. Since, the converse is also true, we get

$$\left\{ \begin{array}{c} (M^2, g) \hookrightarrow \mathbb{E}(\kappa, \tau) \\ \text{of mean curvature } H \end{array} \right\}$$

$$\Updownarrow$$

$$\left\{ \begin{array}{c} M^2 \text{ is a simply connected } \mathrm{Spin}^c \text{ surface} \\ \text{carrying a spinor field } \varphi \text{ of constant norm} \\ \text{satisfying } D^M\varphi = H\varphi - i\tau\overline{\varphi}. \\ \text{The curvature of the auxiliary line bundle} \\ \text{is given by } \ i\Omega^M(e_1, e_2) = -i(\kappa - 4\tau^2) < \varphi, \frac{\overline{\varphi}}{|\varphi|^2} > . \end{array} \right\}.$$

Theorem 7.3 *(**A Lawson type correspondence**). Every simply connected surface minimal in Nil_3 is isometric to a simply connected surface immersed into $\mathbb{H}^2 \times \mathbb{R}$ of constant mean curvature $\frac{1}{2}$.*

Proof. From Proposition 7.1, we have

$$\left\{ \begin{array}{c} (M^2,g) \hookrightarrow \mathbb{H}^2\times\mathbb{R} = \mathbb{E}(-1,0) \\ \text{of mean curvature } H=\frac{1}{2} \end{array} \right\}$$

$$\Updownarrow$$

$$\left\{ \begin{array}{c} M^2 \text{ is a simply connected Spin}^c \text{ surface} \\ \text{carrying a spinor field } \varphi \text{ of constant norm} \\ \text{satisfying } D^M\varphi = \frac{1}{2}\varphi. \\ \text{The curvature of the auxiliary line bundle} \\ \text{is given by } \ i\Omega^M(e_1,e_2) = i < \varphi, \frac{\overline{\varphi}}{|\varphi|^2} > . \end{array} \right\}. \tag{14}$$

From the remark above, we have also,

$$\left\{ \begin{array}{c} (M^2,g) \hookrightarrow \mathrm{Nil}_3 = \mathbb{E}(0,\frac{1}{2}) \\ \text{of mean curvature } H=0 \end{array} \right\}$$

$$\Updownarrow$$

$$\left\{ \begin{array}{c} M^2 \text{ is a simply connected Spin}^c \text{ surface} \\ \text{carrying a spinor field } \Phi \text{ of constant norm} \\ \text{satisfying } D^M\Phi = -i\frac{1}{2}\overline{\Phi}. \\ \text{The curvature of the auxiliary line bundle} \\ \text{is given by } \ i\Omega^M(e_1,e_2) = i < \Phi, \frac{\overline{\Phi}}{|\Phi|^2} > . \end{array} \right\}. \tag{15}$$

To prove the correspondence between minimal surfaces in Nil_3 and surfaces in $\mathbb{H}^2\times\mathbb{R}$ of constant mean curvature $\frac{1}{2}$, we have to prove that Systems (14) and (15) are equivalent. In fact, considering System (14), we define $\Phi = \varphi_+ + i\varphi_-$ which satisfies System (15). Conversely, having System (15), we define $\varphi = \Phi_+ + i\Phi_-$ which satisfies System (14).

Remarks 7.4 *1. The Lawson type correspondence between simply connected constant mean curvature surfaces can be done for all $\mathbb{E}(\kappa,\tau)$ [NR11].*

2. *The Lawson type correspondence between simply connected surfaces in $\mathbb{E}(\kappa,\tau)$ has been proved by B. Daniel with another proof [Dan07].*

3. *The manifolds $\mathbb{E}(\kappa,\tau)$ are also* Spin *manifolds. But, using* Spin *structures on $\mathbb{E}(\kappa,\tau)$, we cannot prove this Lawson type corrrespondence because the* Spin *structure on $\mathbb{E}(\kappa,\tau)$ does not carry a natural spinor field (like a parallel or a Killing spinor field).*

Acknowledgements

These notes are based on a mini-course given at the Max Planck Institute for Mathematics in Bonn, Germany. I would like to thank Nadine Grosse, Julien Roth and Georges Habib for a careful reading of these notes.

References

[Atiy-Sing68] M. Atiyah and I. Singer, *The index of elliptic operators I*, Ann. of Math., 87 (1968), 484-530.

[Atiy78] M. Atiyah, N. Hitchin and I. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London A 362 (1978), 425-461.

[Bär93] C. Bär, *Real Killing spinors and holonomy*, Commun. Math. Phys. 154 (1993), 525-576.

[BGM05] C. Bär, P. Gauduchon and A. Moroianu, *Generalized cylinders in semi-Riemannian* Spin *manifolds*, Math. Z. 249 (2005), 545-580.

[BHMM] J.P. Bourguignon, O. Hijazi, J.L. Milhorat and A. Moroianu, *A spinorial approach to Riemannian and conformal geometry*, Monograph (In preparation).

[Dan07] B. Daniel, *Isometric immersions into* 3*-dimensional homogeneous manifolds*, Comment. Math. Helv. 82 (2007), 87-131.

[Dan09] ———, *Isometric immersions into* $\mathbb{S}^n \times \mathbb{R}$ *and* $\mathbb{H}^n \times \mathbb{R}$ *and applications to minimal surfaces*, Trans. Amer. Math. Soc. 361 (2009), 6255-6282.

[DHM] B. Daniel, L. Hauswirth and P. Mira, *Constant mean curvature surfaces in homogeneous 3-manifolds*, in preparation.

[Don96] S.K. Donaldson, *The Seiberg-Witten equations and* 4*-manifold topology*, Bull. Amer. Math. Soc 33 (1996), 45-70.

[Fri80] T. Friedrich, *Der erste Eigenwert des Dirac-operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung,* Math. Nach. 97 (1980), 117-146.

[Fri98] ———, *On the spinor representation of surfaces in Euclidean 3-spaces*, J. Geom. Phys. 28 (1998), 143-157.

[Fri00] ———, *Dirac operator's in Riemannian Geometry*, Graduate studies in mathematics, Volume 25, Americain Mathamatical Society, 2000.

[Gin09] N. Ginoux, *The Dirac spectrum*, Lect. Notes in Math. 1976, Springer 2009.

[GL98] M.J. Gursky and C. LeBrun, *Yamabe invariants and* Spin^c *structures*, Geom. Func. An. 8 (1998), 965-977.

[Ha-Na10] G. Habib and R. Nakad, *The energy-momentum tensor on low dimensional* Spin^c *manifolds*, submitted paper, 2010.

[Hij95] O. Hijazi, *Lower bounds for the eigenvalues of the Dirac operator*, J. Geom. Phys. 16 (1995) 27-38.

[Hij01] ———, *Spertral properties of the Dirac operator and geometrical structures*, Proceedings of the summer school on geometric methods in quantum field theory, Villa de Leyva, Colombia, July 12-30, 1999, World Scientific 2001.

[HMZ01a] O. Hijazi, S. Montiel and X. Zhang, *Dirac operator on embedded hypersurfaces*, Math. Res. Lett. 8 (2001), 195-208.

[HMZ01b] ———, *Eigenvalues of the Dirac operator on manifolds with boundary*, Comm. Math. Phys., 221 (2001), 255-265.

[HMR02] O. Hijazi, S. Montiel, and S. Roldán, *Eigenvalue boundary problems for the Dirac operator*, Commun. Math. Phys. 231 (2002), 375-390.

[HMU06] O. Hijazi, S. Montiel and F. Urbano, Spin^c *geometry of Kähler manifolds and the Hodge Laplacian on minimal Lagrangian submanifolds*, Math. Z. 253, Number 4 (2006) 821-853.

[Kir86] K.D. Kirchberg, *An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature*, Ann. Glob. Anal. Geom. 3 (1986), 291-325.

[KM94] P. Kronheimer and T. Mrowka, *The genus of embedded surfaces in the projective plane* Math. Res. Lett 1 (1994), 797-808.

[LM89] H.B. Lawson and M.L. Michelson, *Spin Geometry*, Princeton University press, Princeton, New Jersey, 1989.

[LeB95] C. LeBrun, *Einstein metrics and Mostov rigidity*, Math. Res. Lett. 2 (1995), 1-8.

[LeB96] ———, *4-manifolds without Einstein metrics*, Math. Res. Lett. 3 (1996), 133-147.

[Lich63] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris 257 (1963), 7-9.

[Mor05] B. Morel, *Surfaces in* $\mathbb{S}^3$ *and* $\mathbb{H}^3$ *via spinors*, Actes du séminaire de théorie spectrale et géométrie, Institut Fourier, Grenoble 23 (2005), 9-22.

[Mon05] S. Montiel, *Using spinors to study submanifolds*, Roma 2004-Nancy 2005.

[Moro97] A. Moroianu, *Parallel and Killing spinors on* Spin^c *manifolds*, Commun. Math. Phys. 187 (1997), 417-428.

[Moro99] ———, *Kähler manifolds with small eigenvalues of the Dirac operator and a conjecture of Lichnerowicz*, Ann. Inst. Fourier 49 (1999), 1637-1659.

[Moro03] ———, *Kähler geometry*, a course given in 2003 at the University of Hamburg.

[Nak11a] R. Nakad, *The energy-momentum tensor on* Spin^c *manifolds*, IJGMMP Vol. 8, No. 2, 2011.

[Nak11b] ———-, *Sous-variétés spéciales des variétés spinorielles complexes*, Ph.D. thesis, Institut Élie Cartan, 2011.

[NR11] R. Nakad and J. Roth, *Hypersurfaces of Spin*^{c} *manifolds and Lawson type correspondence*, submitted paper, 2011.

[Sei-Wit94] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in* $N = 2$ *supersymmetric QCD*, Nuclear Phys. B 431 (1994), 484-550.

[Wit94] E. Witten, *Monopoles and four-manifolds*, Math. Res. Lett. 1 (1994), 769-796.